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Introduction

This book deals with the application of linear programming to firm decision making. In particular, an important resource allocation problem that often arises in actual practice is when a set of inputs, some of which are limited in supply over a particular production period, is to be utilized to produce, using a given technology, a mix of products that will maximize total profit. While a model such as this can be constructed in a variety of ways and under different sets of assumptions, the discussion that follows shall be limited to the linear case, i.e. we will consider the short-run static profit-maximizing behavior of the multiproduct, multifactor competitive firm that employs a fixed-coefficients technology under certainty (Dorfman 1951, 1953; Naylor 1966).

How may we interpret the assumptions underlying this **profit maximization model**?

1) All-around **perfect competition** – the prices of the firm's product and variable inputs are given.
2) The firm employs a **static model** – all prices, the technology, and the supplies of the fixed factors remain constant over the production period.
3) The firm operates under conditions of **certainty** – the model is deterministic in that all prices and the technology behave in a completely systematic (nonrandom) fashion.
4) All factors and products are **perfectly divisible** – fractional (noninteger) quantities of factors and products are admissible at an optimal feasible solution.
5) The character of the firm's **production activities,** which represent specific ways of combining fixed and variable factors in order to produce a unit of output (in the case where the firm produces a single product) or a unit of an individual product (when the number of activities equals or exceeds the number of products), is determined by a set of technical decisions internal to the firm. These input activities are:
   a) **independent** in that no interaction effects exist between activities;
   b) **linear**, i.e. the input/output ratios for each activity are constant along with returns to scale (if the use of all inputs in an activity increases by

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a fixed amount, the output produced by that activity increases by the same amount);

c) **additive**, e.g. if two activities are used simultaneously, the final quantities of inputs and outputs will be the arithmetic sums of the quantities that would result if these activities were operated separately. In addition, total profit generated from all activities equals the sum of the profits from each individual activity; and
d) **finite** – the number of input activities or processes available for use during any production period is limited.

6) All structural relations exhibit **direct proportionality** – the objective function and all constraints are linear; unit profit and the fixed-factor inputs per unit of output for each activity are directly proportional to the level of operation of the activity (thus, marginal profit equals average profit).
7) The firm's objective is to **maximize total profit** subject to a set of structural activities, fixed-factor availabilities, and nonnegativity restrictions on the activity levels. Actually, this objective is accomplished in two distinct stages. First, a technical optimization problem is solved in that the firm chooses a set of production activities that requires the minimum amount of the fixed and variable inputs per unit of output. Second, the firm solves the aforementioned constrained maximum problem.
8) The firm operates in the **short run** in that a certain number of its inputs are fixed in quantity.

Why is this linear model for the firm important? It is intuitively clear that the more sophisticated the type of capital equipment employed in a production process, the more inflexible it is likely to be relative to the other factors of production with which it is combined. That is, the machinery in question must be used in fixed proportions with regard to certain other factors of production (Dorfman 1953, p. 143). For the type of process just described, no factor substitution is possible; a given output level can be produced by one and only one input combination, i.e. the inputs are **perfectly complementary**. For example, it is widely recognized that certain types of chemical processes exhibit this characteristic in that, to induce a particular type of chemical reaction, the input proportions (coefficient) must be (approximately) fixed. Moreover, mechanical processes such as those encountered in cotton textile manufacturing and machine-tool production are characterized by the presence of this **limitationality**, i.e. in the latter case, constant production times are logged on a fixed set of machines by a given number of operators working with specific grades of raw materials.

For example, suppose that a firm produces three types of precision tools (denoted x_1, x_2, and x_3) made from high-grade steel. Four separate production operations are used: casting, grinding, sharpening, and polishing. The set of input–output coefficients (expressed in minutes per unit of output), which describe the firm's technology (the firm's stage one problem, as alluded to above, has been solved) is presented in Table 1.1. (Note that each of the three columns represents a separate input activity or process.)

Additionally, capacity limitations exist with respect to each of the four production operations in that upper limits on their availability are in force. That is, per production run, the firm has at its disposal 5000 minutes of casting time, 3000 minutes of grinding time, 3700 minutes of sharpening time, and 2000 minutes of polishing time. Finally, the unit profit values for tools x_1, x_2, and x_3 are \$22.50, \$19.75, and \$26.86, respectively. (Here these figures each depict unit revenue less unit variable cost and are computed before deducting fixed costs. Moreover, we are tacitly assuming that what is produced is sold.) Given this information, it is easily shown that the optimization problem the firm must solve (i.e. the stage-two problem mentioned above) will look like (1.1):

$$
\begin{aligned}
max f &= 22.50x_1 + 19.75x_2 + 26.86x_3 \ s.t. \ (\text{subject to}) \\
&13x_1 + 10x_2 + 16x_3 \le 5000 \\
&12x_1 + 8x_2 + 20x_3 \le 3000 \\
&8x_1 + 4x_2 + 9x_3 \le 3700 \\
&5x_1 + 4x_2 + 6x_3 \le 2000 \\
&x_1, x_2, x_3 \ge 0.
\end{aligned} \tag{1.1}
$$

How may we rationalize the structure of this problem? First, the **objective function *f* represents total profit**, which is the sum of the individual (gross) profit contributions of the three products, i.e.

$$
\begin{aligned}
\text{total profit} &= \sum_{j=1}^{3} \left(\text{total profit from } x_j \text{ sales}\right) \\
&= \sum_{j=1}^{3} \left(\text{unit profit from } x_j \text{ sales}\right) \left(\text{number of units of } x_j \text{ sold}\right)
\end{aligned}
$$

Table 1.1 Input–output coefficients.

Tools			
x_1	x_2	x_3	**Operations**
13	10	16	Casting
12	8	20	Grinding
8	4	9	Sharpening
5	4	6	Polishing

Next, if we consider the first **structural constraint** inequality (the others can be interpreted in a similar fashion), we see that total casting time used per production run cannot exceed the total amount available, i.e.

$$\begin{aligned}\text{total casting time used} &= \sum_{j=1}^{3}\left(\text{total casting time used by } x_j\right)\\ &= \sum_{j=1}^{3}\left(\text{casting time used per unit of } x_j\right)\\ &\quad\left(\text{number of units of } x_j \text{ produced}\right) \le 5000.\end{aligned}$$

Finally, the **activity levels** (product quantities) x_1, x_2, and x_3 are nonnegative, thus indicating that the production activities are **nonreversible**, i.e. the fixed inputs cannot be created from the outputs.

To solve (1.1) we shall employ a specialized computational technique called the *simplex method*. The details of the simplex routine, as well as its mathematical foundations and embellishments, will be presented in Chapters 2–5. Putting computational considerations aside for the time being, the types of information sets that the firm obtains from an optimal solution to (1.1) can be characterized as follows. The **optimal product mix** is determined (from this result management can specify which product to produce in positive amounts and which ones to omit from the production plan) as well as the **optimal activity levels** (which indicate the exact number of units of each product produced). In addition, **optimal resource utilization** information is also generated (the solution reveals the amounts of the fixed or scarce resources employed in support of the optimal activity levels) along with the **excess (slack) capacity** figures (if the total amount available of some fixed resource is not fully utilized, the optimal solution indicates the amount left idle). Finally, the **optimal dollar value of total profit** is revealed.

Associated with (1.1) (hereafter called the **primal problem)** is a symmetric problem called its **dual**. While Chapter 6 presents duality theory in considerable detail, let us simply note without further elaboration here that the dual problem deals with the internal valuation (pricing) of the firm's fixed or scarce resources. These (nonmarket) prices or, as they are commonly called, **shadow prices** serve to signal the firm when it would be beneficial, in terms of recouping *forgone profit* (since the capacity limitations restrict the firm's production and thus profit opportunities) to acquire additional units of the fixed factors. Relative to (1.1), the dual problem appears as

$$\begin{aligned}&min\, g = 5000u_1 + 3000u_2 + 3700u_3 + 2000u_4 \quad s.t.\\ &\qquad 13u_1 + 12u_2 + 8u_3 + 5u_4 \ge 22.50\\ &\qquad 10u_1 + 8u_2 + 4u_3 + 4u_4 \ge 19.75\\ &\qquad 16u_1 + 20u_2 + 9u_3 + 6u_4 \ge 26.86\\ &\qquad u_1, u_2, u_3, u_4 \ge 0,\end{aligned} \tag{1.2}$$

where the dual variables $u_1, \ldots, u_4$ are the shadow prices associated with the primal capacity constraints.

What is the interpretation of the form of this dual problem? First, the objective g depicts the **total imputed (accounting) value of the firm's fixed resources**, i.e.

$$\begin{aligned}&\text{total imputed value of all fixed resources}\\&=\sum_{i=1}^{4}(\text{total imputed value of the } i\text{th resource})\\&=\sum_{i=1}^{4}\ (\text{number of units of the } i\text{th resource available})\\&\quad(\text{shadow price of the } i\text{th resource}).\end{aligned}$$

Clearly, the firm must make the value of this figure as small as possible. That is, it must *minimize forgone profit*. Next, looking to the first **structural constraint** inequality in (1.2) (the rationalization of the others follows suit), we see that the total imputed value of all resources going into the production of a unit of x_1 cannot fall short of the profit per unit of x_1, i.e.

$$\begin{aligned}&\text{total imputed value of all resources per unit of } x_1\\&=\sum_{i=1}^{4}(\text{imputed value of the } i\text{th resource per unit of } x_1)\\&=\sum_{i=1}^{4}(\text{number of units of the } i\text{th resource per unit of } x_1)\\&\quad(\text{shadow price of the } i\text{th resource}) \geq 22.50.\end{aligned}$$

Finally, as is the case for any set of prices, the shadow prices $u_1, \ldots, u_4$ are all nonnegative.

As will become evident in Chapter 6, the dual problem does not have to be solved explicitly; its optimal solution is obtained as a byproduct of the optimal solution to the primal problem (and vice versa). What sort of information is provided by the optimal dual solution? The **optimal (internal) valuation of the firm's fixed resources** is exhibited (from this data the firm can discern which resources are in excess supply and which ones are "scarce" in the sense that total profit could possibly be increased if the supply of the latter were augmented) along with the **optimal shadow price configuration** (each such price indicates the increase in total profit resulting from a one unit increase in the associated fixed input). Moreover, the **optimal (imputed) value of inputs** for each product is provided (the solution indicates the imputed value of all fixed resources entering into the production of a unit of each of the firm's outputs) as well as the **optimal accounting loss figures** (here, management is provided with information pertaining to the amount by which the imputed value of all resources used to produce a unit of some product exceeds the unit profit level for the same). Finally, the **optimal imputed value of all fixed resources** is determined. Interestingly enough, this quantity equals the optimal dollar value of total profit obtained from the primal problem, as it must at an optimal feasible solution to the primal-dual pair of problems.

In the preceding model we made the assumption that the various production activities were technologically independent. However, if we now assume that they are **technologically interdependent** in that each product can be produced by employing more than one process, then we may revise the firm's objective to one where a set of production quotas are to be fulfilled at minimum cost. By invoking this assumption we may construct what is called a **joint production model**.

As far as a full description of this type of production program is concerned, let us frame it in terms of the short-run static cost-minimizing behavior of a multiproduct, multifactor competitive firm that employs a fixed-coefficients technology. How can we interpret the assumptions given in support of this model?

1) **Perfect competition** in the factor markets – the prices of the firm's primary and shadow inputs are given.
2) The firm employs a **static model** – all prices, the technology, and the output quotas remain constant over the production period.
3) The firm operates under conditions of **certainty** – the model is deterministic in that all prices and the technology behave in a completely systematic (nonrandom) fashion.
4) All factors and products are **perfectly divisible** – fractional quantities of factors and products are admissible at an optimal feasible solution.
5) The character of the firm's **production activities**, which now represent ways of producing a set of outputs from the application of one unit of a primary input, is determined by a set of technical decisions internal to the firm. These output activities are:
   a) **independent** in that no interaction effects exist among activities;
   b) **linear**, i.e. the output/input ratios for each activity are constant along with the input response to an increase in outputs (if the production of all outputs in an activity increases by a fixed amount, then the input level required by the process must increase by the same amount);
   c) **additive**, e.g. if two activities are used simultaneously, the final quantities of inputs and outputs will be the arithmetic sums of the quantities which would result if these activities were operated separately. Moreover, the total cost figure resulting from all output activities equals the sum of the costs from each individual activity; and
   d) **finite** – the number of output activities or processes available for use during any production period is limited.
6) All structural relations exhibit **direct proportionality** – the objective function and all constraints are linear; unit cost and the fixed-output per unit of

input values for each activity are directly proportional to the level of operation of the activity. (Thus marginal cost equals average cost.)

7) The firm's objective is to **minimize total cost** subject to a set of structural activities, fixed output quotas, and nonnegativity restrictions on the activity levels. This objective is also accomplished in two stages, i.e. in stage one a technical optimization problem is solved in that the firm chooses a set of output activities which yield the maximum amounts of the various outputs per unit of the primary factors. Second, the firm solves the indicated constrained minimization problem.
8) The **short-run** prevails in that the firm's minimum output requirements are fixed in quantity.

For the type of output activities just described, no output substitution is possible; producing more of one output and less of another is not technologically feasible, i.e. the outputs **are perfectly complementary** or **limitational** in that they must all change together.

As an example of the type of model just described, let us assume that a firm employs three grades of the primary input labor (denoted x_1, x_2, and x_3) to produce four separate products: chairs, benches, tables, and stools. The set of output–input coefficients (expressed in units of output per man-hour) which describe the firm's technology appears in Table 1.2. (Here each of the three columns depicts a separate output activity.) Additionally, output quotas exist with respect to each of the four products in that lower limits on the number of units produced must not be violated, i.e. per production run, the firm must produce at least eight chairs, four benches, two tables, and eight stools. Finally, the unit cost coefficients for the labor grades x_1, x_2, and x_3 are \$8.50, \$9.75, and \$9.08, respectively. (Each of these latter figures depicts unit primary resource cost plus unit

Table 1.2 Output–input coefficients.

Grades of Labor			
x_1	x_2	x_3	**Outputs**
$\frac{1}{16}$	$\frac{1}{14}$	$\frac{1}{18}$	Chairs
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	Benches
$\frac{1}{20}$	$\frac{1}{25}$	$\frac{1}{30}$	Tables
$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$	Stools

shadow input cost.) Given this information, the firm's optimization problem may be written as:

$$
\begin{aligned}
& min\, f = 8.50x_1 + 9.75x_2 + 9.08x_3 \qquad s.t. \\
& \quad \frac{1}{16}x_1 + \frac{1}{14}x_2 + \frac{1}{18}x_3 \geq 8 \\
& \quad \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{6}x_3 \geq 4 \\
& \quad \frac{1}{20}x_1 + \frac{1}{25}x_2 + \frac{1}{30}x_3 \geq 2 \\
& \quad \frac{1}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{6}x_3 \geq 8 \\
& \quad x_1, x_2, x_3 \geq 0.
\end{aligned}
\tag{1.3}
$$

How may we interpret the structure of this problem? First, the **objective function f represents total cost**, expressed as the sum of the individual cost contributions of the various labor grades and shadow factors, i.e.

$$
\begin{aligned}
\text{total cost} &= \sum_{j=1}^{3} (\text{total cost of all resources associated with the } j\text{th output activity}) \\
&= \sum_{j=1}^{3} (\text{total cost of all resources per unit of the } j\text{th output activity}) \\
&\quad (\text{number of units of } x_j \text{ used}).
\end{aligned}
$$

Next, if we concentrate on the first **structural constraint** inequality (the others are interpreted in like fashion), we see that the total number of chairs produced per production run cannot fall short of the total number required, i.e.

$$
\begin{aligned}
& \text{total number of chairs produced} \\
& \quad = \sum_{j=1}^{3} (\text{number of chairs produced using } x_j) \\
& \quad = \sum_{j=1}^{3} (\text{number of chairs produced per unit of } x_j) \\
& \qquad (\text{number of units of } x_j \text{ used}) \geq 8.
\end{aligned}
$$

Finally, the **activity levels** (units of the various primary-input grades employed) x_1, x_2, and x_3 are all nonnegative.

The information content of an optimal feasible solution to (1.3) can be characterized as follows. The **optimal primary-factor or labor-grade mix** is defined (from this result management can resolve the problem of which grades of labor to use in positive amounts and which ones not to employ) as well as the **optimal output activity levels** (the exact number of units of each labor grade utilized is indicated). Moreover, the **optimal output configuration** is decided (the solution reveals the amounts of each of the outputs produced) along with the set of **overproduction figures** (which give the amounts by which any of the production quotas are exceeded). Finally, the decision makers are provided with the **optimal dollar value of total cost**.

As was the case with (1.1), the primal problem (1.3) has associated with it a symmetric dual problem which deals with the assessment of the opportunity costs associated with fulfilling the firm's output quotas. These costs or, more properly, **marginal (imputed or shadow) costs**, are the dual variables which serve to inform the firm of the "potential" cost reduction resulting from a unit decrease in the *i*th (minimum) output requirement (since these production quotas obviously limit the firm's ability to reduce total cost by employing fewer inputs). Using (1.3), the dual problem has the form

$$\begin{aligned} max\, g = 8u_1 + 4u_2 + 2u_3 + 8u_4 \qquad s.t.& \\ \frac{1}{16}u_1 + \frac{1}{4}u_2 + \frac{1}{20}u_3 + \frac{1}{4}u_4 \le 8.50& \\ \frac{1}{14}u_1 + \frac{1}{4}u_2 + \frac{1}{25}u_3 + \frac{1}{3}u_4 \le 9.75& \\ \frac{1}{18}u_1 + \frac{1}{6}u_2 + \frac{1}{30}u_3 + \frac{1}{6}u_4 \le 9.08& \\ u_1, u_2, u_3, u_4 \ge 0,& \end{aligned} \tag{1.4}$$

where the dual variables $u_1, \ldots, u_4$ are the marginal (imputed) costs associated with the set of (minimum) output structural constraints.

What is the economic meaning of the form of this dual problem? First, the objective g represents **the total imputed cost of the firm's minimum output requirements**,

$$\begin{aligned} &\text{total imputed cost of all output quotas} \\ &= \sum_{i=1}^{4} (\text{total imputed cost of the } i\text{th output quota}) \\ &= \sum_{i=1}^{4} (i\text{th output quota})\,(\text{marginal cost of the } i\text{th output}). \end{aligned}$$

Clearly the firm must make the value of this figure as large as possible, i.e. the firm seeks to maximize its *total potential cost reduction*. Next, upon examining the first **structural constraint** inequality in (1.4) (the other two are interpreted in a similar fashion) we see that the total imputed cost of the outputs produced by operating the *j*th activity at the unit level cannot exceed the total cost of all inputs per unit of the *j*th activity, i.e.

$$\begin{aligned}
&\text{total imputed cost of all outputs per unit of } x_1 \\
&= \sum_{i=1}^{4} (\text{ imputed value of the } i\text{th output per unit of } x_1) \\
&= \sum_{i=1}^{4} (\text{number of units of the } i\text{th output per unit of } x_1) \\
&(\text{marginal cost of the } i\text{th output}) \le 8.50.
\end{aligned}$$

Finally, the **marginal cost** figures $u_1, \ldots, u_4$ are all required to be nonnegative.

How may we interpret the data sets provided by an optimal feasible solution to this dual problem?

The set of **optimal imputed costs of the output quotas** is rendered, and from this information the firm can determine which production quotas are fulfilled and which ones are exceeded.

The **marginal (imputed) cost** configuration is determined. Each such figure reveals the potential cost reduction to the firm if the associated output quota is reduced by one unit.

Furthermore, the **optimal (imputed) value of outputs produced** for each primary factor grade is computed. Here we obtain data on the imputed cost of all outputs produced by each primary factor.

The **optimal accounting loss figures** are calculated. Here management is apprised of the amount by which the total of all resources per unit of activity *j* exceeds the total imputed cost of the outputs produced by running activity *j* at the unit level.

The **total imputed cost of all output requirements** is determined. Here, too, the optimal values of the primal and dual objectives are equal.

While it is important to obtain the information contained within an optimal solution to the primal and dual problems, additional sets of calculations that are essential for purposes of determining the *robustness* of, say, the optimal primal solution are subsumed under the heading of *postoptimality analysis*. For example, we can characterize the relevant types of postoptimality computations as follows:

a) *Sensitivity analysis* (Chapter 8) involves the introduction of discrete changes in any of the unit profit, input–output, or capacity values, i.e. these quantities are altered (increased or decreased) in order to determine the extent to which the original problem may be modified without violating the feasibility or optimality of the original solution

b) *Analyzing structural changes* (Chapter 9) determines the effect on an optimal basic feasible solution to a given linear programming problem of the addition or deletion of certain variables or structural constraints
c) *Parametric analysis* (Chapter 10) generates a sequence of basic solutions, which, in turn, become optimal, one after the other, as any or all of the unit profit coefficients or capacity restrictions or components of a particular activity vary continuously in some prescribed direction.

Once the reader has been exposed to parametric programming techniques, it is but a short step to the application of the same (see Chapter 11) in the derivation of the following:

- Supply function for the output of an activity
- Demand function for a variable input
- Marginal (net) revenue productivity function for a fixed input
- Marginal cost function for an activity
- Marginal and average productivity functions for a fixed input along with the marginal and average cost functions for the firm's output

Next, with reference to the cost minimization objective within the joint production model, we shall again employ the technique of parametric programming to derive the total, marginal, and average cost functions for a joint product. In addition, the supply function for the same is also developed.

In order to set the stage for the presentation of the theory and economic applications of linear programming, Chapter 2 discusses the rudiments of matrix algebra, the evaluation of determinants, elementary row operations, matrix inversion, vector algebra and vector spaces, simultaneous linear equation systems, linear dependence and rank, basic solutions, convex sets, and n-dimensional geometry and convex cones.